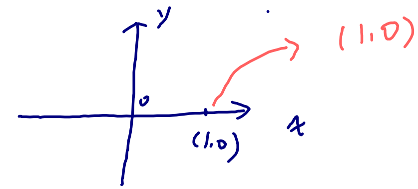
Math2010 B Tutorial 9.

Outline :

- Higher order derivatives in polar coord.

- Implicit differentiation.

e.g. Let $f(x,y) = \underline{\frac{x^2}{2} - y}$, $x, y \in \mathbb{R}$



Find $f_{r\theta}$ at $(x,y) = (1,0)$

$$f_{r\theta} = \frac{\partial}{\partial \theta}(f_r)$$

$$\left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right. \Rightarrow \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\Downarrow$$

$$\left\{ \begin{array}{l} \frac{\partial x}{\partial r} = \cos\theta \\ \frac{\partial y}{\partial r} = \sin\theta \end{array} \right.$$

Method 1: $f(x,y) = f(r,\theta) \Longrightarrow f_{r\theta}$

$$\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$$

2: regard x. y as function of (r,θ)

$$f_r = f_x \cdot \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} \qquad f_x = x \qquad f_y = -1$$

$$= f_x \cdot \cos\theta + f_y \sin\theta$$

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e.g $\underline{f_r = x \cdot \cos\theta + (-1)\cdot\sin\theta}$

$$f_{r\theta} = \frac{\partial}{\partial\theta}(f_r) = \frac{\partial}{\partial\theta}(x\cos\theta - \sin\theta)$$

$$= \left(\frac{\partial}{\partial\theta}x\right)\cdot\cos\theta + x(-\sin\theta) - \cos\theta$$

$$= -r\sin\theta\cos\theta - x\sin\theta - \cos\theta$$

$f_{r\theta}(1,0) = -1$.

Higher order derivative in Polar Coord.

Formula:

Let $\mathbb{R}^2 \underset{\text{open}}{\supset} \Omega \xrightarrow{f} \mathbb{R}$ be of class C^2 i.e f_{xx}. f_{yy} f_{yx}. f_{xy} continuous

suppose f is expressed in rectangular coord (x, y).

So that we can easily calculate

f_x. f_y f_{xx}, $f_{xy} = f_{yx}$, f_{yy}.

We can find partial derivatives in polar coord (r, θ) by chain rule.

① $f_r = \cos\theta \, f_x + \sin\theta \, f_y$ ← Chain rule.

② $f_\theta = -r\sin\theta \, f_x + r\cos\theta \, f_y$

③ $f_{rr} = \cos^2\theta \, f_{xx} + \sin 2\theta \, f_{xy} + \sin^2\theta \, f_{yy}$

④ $f_{r\theta} = \frac{\partial f_r}{\partial \theta} = -\sin\theta \, f_x + \cos\theta \, f_y - \frac{r}{2}\sin 2\theta \, f_{xx} + r\cos\theta \, f_{xy} + \frac{r}{2}\sin 2\theta \, f_{yy}$

⑤ $f_{\theta\theta} = -r\cos\theta \, f_x - r\sin\theta \, f_y + r^2\sin^2\theta \, f_{xx} - r^2\sin 2\theta \, f_{xy} + r^2\cos^2\theta \, f_{yy}$.

Implicit Differentiation

e.g. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x,y,z)=(1,2,1)$ in $\boxed{y^{xz+1}=4}$

$z = z(x,y)$

x, y variables

$z(x,y)$.

Sol:

$$y^{xz+1} = 4$$

$\frac{\partial}{\partial x}$ on both sides,

$$\frac{\partial}{\partial x}\left(y^{(xz+1)}\right) = 0$$

$\frac{\partial}{\partial x}\left(a^{(xz+1)}\right)$

w/ $a=y$

$$\|$$

$$\underline{\left(y^{xz+1}\ln y\right)} \cdot \frac{\partial}{\partial x}(xz+1) = 0$$

$$\left(y^{xz+1}\ln y\right)\cdot\left(z + x\cdot\frac{\partial z}{\partial x}\right) = 0 \quad *$$

(recall $\frac{d}{dx}a^x \overset{?}{=} a^x \ln a$

$\frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot \ln a$

$= a^x \cdot \ln a$)

Putting $(x,y,z) = (1,2,1)$

$$\Rightarrow 4\cdot\ln 2\cdot\left(1+\frac{\partial z}{\partial x}\right) = 0 \quad \Rightarrow \quad \left.\frac{\partial z}{\partial x}\right|_{(x,y,z)=(1,2,1)} = -1$$

$\frac{\partial}{\partial y}$ on both sides of $y^{xz+1} = 4$

$\longrightarrow y^{xz+1} \frac{\partial}{\partial y}\left(\ln y \; (xz+1) \right) = 0$

$y^{xz+1}\left(\frac{1}{y}(xz+1) + \ln y \cdot x \frac{\partial z}{\partial y} \right) = 0$

Putting $(x, y, z) = (1, 2, 1)$

$4 \left(\frac{1}{2} (1 \cdot 1 + 1) + \ln 2 \cdot 1 \cdot \left.\frac{\partial z}{\partial y}\right|_{(x,y,z)=(1,2,1)} \right) = 0$

$\Rightarrow \left.\frac{\partial z}{\partial y}\right|_{(x,y,z)=(1,2,1)} = -\frac{1}{\ln 2}$

Remark: $\frac{\partial}{\partial y}\left(y^{xz+1} \right) \neq (xz+1) y^{xz}$ as $xz+1$ depend on y!

$\left(\begin{aligned} & y^{xz+1} = e^{(xz+1)\ln y} \\ & \frac{\partial}{\partial y}\left(y^{xz+1} \right) = e^{(xz+1)\ln y} \cdot \frac{\partial}{\partial y}\left(\ln y \, (xz+1) \right) \end{aligned} \right)$

but $\frac{\partial}{\partial y}\left(\ln y \cdot (xz+1) \right)$

$= \frac{1}{y}(xz+1) + \ln y \left(\frac{\partial}{\partial y}(xz+1) \right)$

$= \frac{1}{y}(xz+1) + \ln y \left(x \frac{\partial z}{\partial y} \right)$

$\underset{\triangle}{y}^{\,x \underset{\triangle}{z} + 1} = e^{(xz+1)\ln y}$

e.g. Consider the following equation

$$\int_0^{x} (t+y)^2 \, dt = \frac{7}{3} \qquad \text{✻}$$

1) slove (✻) if $x=1$.

2) Find $\frac{\partial y}{\partial x}$ in (✻) at $(x,y) = (1,1)$.

Sol: 1) Putting $x=1$

$$\int_0^1 (t+y)^2 \, dt = \frac{7}{3}$$

$$\frac{1}{3}(t+y)^3 \Big|_0^1 = \frac{7}{3}$$

$$(1+y)^3 - y^3 = 7$$

$$(y+2)(y-1) = 0$$

$\therefore$ $y=1$ or -2

2) Simplifying the integral on L. H. S of (*)

$$(x+y)^3 - y^3 = 7 \qquad \frac{\partial y}{\partial x} \quad \text{regard } y$$

$\frac{d}{dx}$ on both sides

$$3(x+y)^2\left(\underset{\frac{dx}{dx}}{1} + \frac{dy}{dx}\right) - 3y^2 \cdot \frac{dy}{dx} = 0$$

$$(x+y)^2 + (x^2+2xy)\frac{dy}{dx} = 0.$$

Putting $(x, y) = (1, 1)$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = -\frac{(1+1)^2}{1^2 + 2\cdot 1\cdot 1} = -\frac{4}{3}$$